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SADDLE POINT THEOREMS AND THE POINT SPECTRUM OF SOME SEMILINEAR ELLIPTIC OPERATORS

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1. Introduction

Critical Point Theory has shown to be a powerful tool for the solution of linear and nonlinear problems in Analysis, both of abstract nature and/or described by ordinary or partial differential equations. Indeed as long as these problems are of variational nature, their solutions are precisely the points x where the derivative $f'(x)$ of a certain differentiable functional f, attached to the problem and defined on a suitable Banach space E, vanishes; that is, the *critical* (or *stationary) points* of f.

While relative minima or maxima yield the most familiar kind of critical points, one is often led to consider *saddle points* of f, that is (strictly speaking) points $x_0 \in E$ such that, for some neighborhood U of $0 \in E$,

$$\begin{cases} f(x_0+v) \leq f(x_0), & v \in U \cap V \\ f(x_0+w) \geq f(x_0), & w \in U \cap W \end{cases}$$

where V, W are complementary subspaces of E. By extension, is often named "saddle point" any critical point of f which - loosely speaking - stems from an essentially different behaviour of f on two complementary subspaces of E.

Fundamental results on the existence and properties of critical points of saddle type for C^1 functionals on Banach spaces have been proved among others by P. H. Rabinowitz, and we recommend in particular his monograph [Ra] for an overview of this subject and of its applications; see also [Kr] and [Pa] for general reference in Critical Point Theory.

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In this lecture (which is based on the paper [C2]), we shall briefly describe how these methods can be used in particular to study the effect that some nonlinear perturbations have upon the spectra of linear elliptic operators acting in a bounded domain Ω of $\mathbb{R}^N$. Precisely, we consider the semilinear elliptic eigenvalue problem

$$(1.1) \qquad \begin{cases} Lu = \mu u + m(x,u)u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\partial\Omega$ is the boundary of Ω and L is the uniformly elliptic operator

$$Lu := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial u}{\partial x_i}) + a_o(x)u$$

with L^∞ coefficients $a_{ij} = a_{ji} \quad (i,j = 1,\ldots,N)$ and a_0, while $m = m(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to be uniformly bounded and (for simplicity) continuous in both variables. Without loss of generality (see [C2]) we can assume that

$$(\text{H0}) \qquad 0 \le m(x,s) \le a$$

for some $a \ge 0$ and all $(x,s) \in \Omega \times \mathbb{R}$. Since $u = 0$ solves (1.1) for all $\mu \in \mathbb{R}$, we look for values of μ (*eigenvalues*) for which there exists a nontrivial solution (an *eigenfunction*) of (1.1). We let Σ denote the *spectrum* of (1.1), that is

$$\Sigma = \{\mu \in \mathbb{R} : \mu \quad \text{is an eigenvalue of (1.1)}\}.$$

As is well known from linear spectral theory, the eigenvalues of the problem

$$(1.2) \qquad \begin{cases} Lu = \mu u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

form an infinite sequence $\mu_1^0 < \mu_2^0 \le \mu_3^0 \le \ldots$ with $\mu_n^0 \to \infty$ as $n \to \infty$; each eigenvalue is repeated as many times as its multiplicity. We let μ_0 be a fixed *higher order* eigenvalue of (1.2) (i.e. $\mu_0 = \mu_k^0$ for some $k > 1$)and ask about the structure of Σ near μ_0.

When m does not depend on s, i.e. $m(x,s) = m(x)$ with $m \in L^\infty$, then (1.1) is itself a linear problem of the same kind of (1.2), except that L is replaced by $\tilde{L}$, $\tilde{L}u = Lu - m(x)u$.

The corresponding spectrum is thus of the same type, i.e. formed by a sequence going off to $+\infty$, and therefore near μ^0 Σ will consist of finitely many points. We shall show on the contrary that when - loosely speaking - m depends on s in a nontrivial way, then locally near μ^0 Σ is an *interval* . In particular, we shall give conditions on m ensuring that for some neighborhood $\mathcal{U}$ of μ_0,

$$]\mu_0 - a, \mu_0[\subset \Sigma \cap \mathcal{U} \subset [\mu_0 - a, \mu_0].$$

We deal with (1.1) by variational methods, and consequently seek its (weak) solutions as critical points of some suitable functional. However, two different points of view can be adopted about (1.1), depending on whether one looks at it as a *constrained* critical point problem or rather as a *free* critical point problem. To be more precise, we let $H_0^1(\Omega)$ be the first Sobolev space on Ω equipped with scalar product and norm

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx \qquad \|u\|^2 = (u, u).$$

A weak solution of (1.1) is an $u \in H_0^1(\Omega)$ such that

$$a(u, v) = \mu \int_\Omega uv \, dx + \int_\Omega m(x, u)uv \, dx \qquad \forall v \in H_0^1(\Omega) \tag{1.3}$$

where

$$a(u, v) = \sum_{i,j=1}^N \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \int_\Omega a_0(x) uv \, dx \tag{1.4}$$

is the Dirichlet form associated with L. Let $Q_0(u) = a(u, u)$ be the corresponding quadratic form; assuming - as we do here and henceforth - that $a_0 \geq 0$ a.e. in Ω, we have $Q_0(u) \geq \mu_0^1 \|u\|^2$ for all $u \in H_0^1(\Omega)$ by the variational characterization of the first eigenvalue of (1.2) ([CH, Chapter 6]). Also set $F(x, t) = \int_0^t m(x, s)s \, ds$ for $(x, t) \in \Omega \times \mathbb{R}$ and define the functionals I and J on $H_0^1(\Omega)$ by the rules

$$J(u) = \int_\Omega F(x, u(x)) \, dx, \qquad I(u) = \frac{1}{2} Q_0(u) - J(u). \tag{1.5}$$

Then (1.3) can be written

$$I'(u)v = \mu \int_\Omega uv\, dx \qquad \forall v \in H_0^1(\Omega) \tag{1.6}$$

where $I'(u)$ stands for the Fréchet derivative of I at the point u. Therefore, finding a solution $u \in H_0^1(\Omega)$ of (1.1) with given L^2 norm $\int_\Omega u^2(x)\, dx = r^2$ is equivalent to finding a *constrained* critical point of I on the manifold

$$M_r = \{u \in H_0^1(\Omega) : \int_\Omega u^2(x)\, dx = r^2\}. \tag{1.7}$$

In this case, μ appears as a *Lagrange multiplier*, and we have to find solutions $u_r \in M_r$ with Lagrange multiplier μ_r near μ_0. On the other hand, we can let μ run as *independent variable* near μ_0 and, setting

$$I_\mu(u) = I(u) - \frac{\mu}{2} \int_\Omega u^2(x)\, dx, \tag{1.8}$$

can write (1.3) as

$$I'_\mu(u)v = I'(u)v - \mu \int_\Omega uv\, dx = 0 \qquad \forall v \in H_0^1(\Omega). \tag{1.9}$$

Following this alternative point of view, we are looking at *free* (nontrivial) critical points of I_μ on $H_0^1(\Omega)$ for μ near μ_0. We shall employ both methods, applying to our concrete problem two different abstract results on the existence of saddle points for a C^1 functional f on a Banach space X (respectively, Theorem A [C1] in Section 2 and Theorem B [Ra] in Section 3).

Our results depend on the assumption that m be small with respect to $d(\mu_0)$, where $d(\mu_0) = dist(\mu_0, \sigma \setminus \{\mu_0\})$ denotes the isolation distance of μ_0 in the spectrum $\sigma = \{\mu_n^0 : n \in \mathbb{N}\}$ of (1.2). Precisely, letting $\underline{\mu} < \mu_0 < \overline{\mu}$ be the eigenvalues of (1.2) nearest to μ_0, we assume at first that

$$a < d(\mu_0) = \min\{\mu_0 - \underline{\mu}, \overline{\mu} - \mu_0\}. \tag{H1}$$

Thus by assumption,

$$\underline{\mu} < \mu_0 - a \quad \text{and} \quad \mu_0 < \overline{\mu} - a.$$

Proposition 1. *Let (H0) and (H1) be satisfied, and suppose that u is a solution of (1.1) corresponding to some $\mu \in]\underline{\mu}, \mu_0 - a[\bigcup]\mu_0, \overline{\mu} - a[$. Then $u = 0$.*

This is a simple consequence of the comparison principle [CH, Chapter 6] for the eigenvalues of linear problems such as (1.2); see [C2].

Therefore, as a first information on Σ near μ^0, we have that

$$\Sigma_0 \equiv \Sigma \bigcap]\underline{\mu}, \overline{\mu} - a[\subset [\mu_0 - a, \mu_0].$$

2. Results by Constrained Critical Point Theory

We now strenghten (H1) to

$$\text{(H2)} \qquad 2a < d(\mu_0).$$

Proposition 2. *Let (H0) and (H2) be satisfied. Then for each $r > 0$, (1.1) possesses an eigenfunction-eigenvalue pair $(u_r, \mu_r) \in H_0^1(\Omega) \times \mathbb{R}$ with $\int_\Omega u_r^2\, dx = r^2$ and*

$$\text{(E)} \qquad \mu_0 - a \le \mu_r \le \mu_0.$$

Proposition 2 is a consequence of the following abstract result [C1]. Let X be a real Banach space, let f be a C^1 functional on X, and let M be a C^1 submanifold of X; also, let $f_M \equiv f|_M$ denote the restriction of f to M. We recall that f is said to satisfy the *Palais-Smale* (PS) *condition* on M if any sequence $(x_n) \subset M$ such that $f_M(x_n)$ is bounded and $f'_M(x_n) \to 0$ contains a convergent subsequence. Moreover, we shall say that a C^1 submanifold M of X (not containing the origin) is *spherelike* if it is radially diffeomorphic to $S = \{x \in X : \|x\| = 1\}$, i.e. (C^1) diffeomorphic to S via the radial projection $R(x) = \dfrac{x}{\|x\|}$, $x \neq 0$. Finally, we recall that $c \in \mathbb{R}$ is a *critical value* of f if $f(x) = c$ for some critical point x.

Theorem A (Constrained Saddle Point Theorem [C1]**).** *Let X be a Banach space, let $f \in C^1(X;\mathbb{R})$, and let M be a C^2 spherelike submanifold of X. Assume that f is bounded below on M and satisfies the (PS) condition on M. Suppose further that $X = V \oplus W$ with $dim V < \infty$, and let α, β be such that*

$$\begin{cases} f(x) \leq \alpha & \text{on } M \cap V \\ f(x) \geq \beta & \text{on } M \cap W. \end{cases} \tag{2.1}$$

Then if $\alpha < \beta$, f has a critical value c on M satisfying

$$\theta \leq c \leq \alpha \tag{2.2}$$

where $f(x) \geq \theta$ on $M \cap (V_0 \oplus W)$, V_0 being a nontrivial subspace of V.

Sketch of the proof of Proposition 2:

Apply Theorem A with $X = H_0^1(\Omega), f = I, M = M_r$ as defined in (1.5) and (1.7). Indeed (see [C1] or [C2]), M_r is a C^2 spherelike submanifold of X and I is bounded below on M_r and satisfies (PS) on M_r. In particular, (H0) implies that

$$0 \leq F(x,t) \leq \frac{a}{2}t^2 \qquad \forall (x,t) \in \Omega \times \mathbb{R} \tag{2.3}$$

and so

$$0 \leq J(u) \leq \frac{a}{2}\int_\Omega u^2(x)\,dx \qquad \forall u \in H_0^1(\Omega). \tag{2.4}$$

Therefore we have

$$I(u) = \frac{1}{2}Q_0(u) - J(u) \geq \frac{1}{2}(\mu_1^0 - a)r^2 \qquad \text{on} \quad M_r. \tag{2.5}$$

Next let V be the orthogonal (in the L^2 sense) sum of the eigenspaces corresponding to all eigenvalues μ of L_0 with $\mu \leq \mu_0$, let V_0 be the eigenspace corresponding to μ_0, and let $W = \{u \in H_0^1(\Omega) : \int_\Omega uv = 0 \quad \forall v \in V\}$. Using the variational characterization of the eigenvalues ([CH, Chapter 6]) and (2.4), we obtain

$$\text{(2.6)} \qquad \begin{cases} I(u) \le \frac{1}{2}\mu_0 r^2 & \text{on } M_r \cap V \\ I(u) \ge \frac{1}{2}(\overline{\mu} - a)r^2 & \text{on } M_r \cap W \\ I(u) \ge \frac{1}{2}(\mu_0 - a)r^2 & \text{on } M_r \cap (V_0 \oplus W). \end{cases}$$

Now (H1) implies that $\mu_0 < \overline{\mu} - a$, and so the condition $\alpha < \beta$ required in Theorem A is satisfied on M_r with $\alpha = \frac{1}{2}\mu_0 r^2, \beta = \frac{1}{2}(\overline{\mu} - a)r^2$. We conclude from Theorem A that I has a critical value c_r on M_r, i.e. there exists $(u_r, \mu_r) \in M_r \times \mathbb{R}$ so that

$$\text{(2.7)} \qquad I(u_r) = c_r, \qquad I'(u_r)v = \mu_r \int_\Omega u_r v \quad \forall v \in H_0^1(\Omega);$$

moreover c_r satisfies the estimate

$$\text{(2.8)} \qquad \frac{1}{2}(\mu_0 - a)r^2 \le c_r \le \frac{1}{2}\mu_0 r^2.$$

Using (2.7), we can also estimate the difference $c_r - \frac{1}{2}\mu_r r^2$ to deduce the corresponding bounds for μ_r, which turn out to be

$$\text{(2.9)} \qquad \mu_0 - 2a \le \mu_r \le \mu_0 + a$$

However, (H2) implies that $\underline{\mu} < \mu_0 - 2a$ and $\mu_0 + a < \overline{\mu} - a$; therefore, using Proposition 1 we infer that μ_r satisfies the improved bounds (E).

When, in addition to the boundedness of $m = m(., s)$, we know more about its behaviour at $s = 0$ and for $|s| \to \infty$, then correspondingly the information about μ_r is richer.

Proposition 3. *Let (H0) and (H2) be satisfied and let $\mu_r (r > 0)$ be as in Proposition 2. Suppose moreover that*

$$\text{(H3)} \qquad \lim_{s \to 0} m(x, s) = m_0, \qquad \lim_{|s| \to \infty} m(x, s) = m_\infty$$

uniformly for $x \in \Omega$. Then $\mu_r \to \mu_0 - m_0$ as $r \to 0$ and $\mu_r \to \mu_0 - m_\infty$ as $r \to \infty$.

Proof (Sketch): It follows from (H0) and (H3) that

$$\frac{2F(x,s)}{s^2} \xrightarrow[s\to 0]{} m_0, \qquad \frac{2F(x,s)}{s^2} \xrightarrow[|s|\to\infty]{} m_\infty \tag{2.10}$$

uniformly for $x \in \Omega$. Now it is just a matter of refining the estimate (2.11) for c_r: indeed, looking at Theorem A we see that $\theta_r \le c_r \le \alpha_r$ whenever $I \le \alpha_r$ on $M_r \cap V$, $I \ge \theta_r$ on $M_r \cap (V_0 \oplus W)$. See [C2] for details.

3. Results by Free Critical Point Theory

Let us collect the informations obtained so far about $\Sigma_0 = \Sigma \bigcap]\underline{\mu}, \overline{\mu} - a[$. We have first seen (Proposition 1) that, under the assumptions (H0) and (H1), $\Sigma_0 \subset [\mu_0 - a, \mu_0]$. Next, reinforcing (H1) to (H2), Proposition 2 shows that (1.1) possesses a one-parameter family $(\mu_r)_{r>0}$ of eigenvalues with $\mu_0 - a \le \mu_r \le \mu_0$ for all $r > 0$; that is,

$$\{\mu_r : r > 0\} \subset \Sigma_0 \subset [\mu_0 - a, \mu_0].$$

Finally by Proposition 3, we have that $\lim_{r\to 0} \mu_r = \mu_0 - m_0$ and $\lim_{r\to\infty} \mu_r = \mu_0 - m_\infty$ if in addition (H3) is satisfied. Evidently $0 \le m_0, m_\infty \le a$; and it follows that if $m_0 \ne m_\infty$, then Σ_0 contains at least two distinct points. It is now natural to ask whether Σ_0 contains an *interval* , and whether in particular, if e.g. $m_0 < m_\infty$, it contains the interval $]\mu_0 - m_\infty, \mu_0 - m_0[$. This is indeed the case:

Proposition 4. *If (H0), (H1) and (H3) hold, and if moreover $m_0 < m_\infty$, then for each $\mu \in]\mu_0 - m_\infty, \mu_0 - m_0[$ there exists a nontrivial solution of (1.1); that is,*

$$]\mu_0 - m_\infty, \mu_0 - m_0[\subset \Sigma_0. \tag{3.1}$$

Proposition 4 is a consequence of the following abstract result [Ra]. Let X be a Banach space; for $r > 0$, we set $B_r = \{x \in X : \|x\| \le r\}$ and $S_r = \{x \in X : \|x\| = r\}$.

Theorem B (Generalized Mountain Pass Theorem [Ra]**).** *Let X be a Banach space and let f be a C^1 functional on X satisfying the (PS) condition. Suppose that $X = \hat{V} \oplus \hat{W}$ with $dim\hat{V} < \infty$. Given $e \in \hat{W}$ with $\|e\| = 1$, set for $R > 0$*

$$(3.2) \qquad Q_R := (B_R \cap \hat{V}) \oplus \{te : 0 \leq t \leq R\}$$

and denote with ∂Q_R the boundary of Q_R relative to the subspace $\hat{V} \oplus \mathbb{R}e$. Assume that there exist $\beta > 0$ and $R > \rho$ such that

$$(3.3) \qquad \begin{cases} f(x) \leq 0 & \text{on} \quad \partial Q_R \\ f(x) \geq \beta & \text{on} \quad S_\rho \cap \hat{W}. \end{cases}$$

Then f has a critical value $c \geq \beta$. In particular, $c > 0$ and so, if $f(0) = 0$, f has a nontrivial *critical point.*

Remark.

i) If $\hat{V} = \{0\}$, then Theorem B reduces to the ordinary Mountain Pass Theorem ([AR]).

ii) Looking at ∂Q_R, it is easy to check that the first condition in (3.3) is satisfied if

a) $f \leq 0$ on $\hat{V}$ and b) $f \leq 0$ on $\{x \in \hat{V} \oplus \mathbb{R}e : \|x\| \geq R\}$.

Sketch of the Proof of Proposition 4:

Apply Theorem B taking $X = H_0^1(\Omega)$ and $f = I_\mu$ as defined in (1.10), with

$$(3.4) \qquad \mu_0 - m_\infty < \mu < \mu_0 - m_0.$$

Moreover, letting V_0, V and W be as in the proof of Proposition 2, we choose $\hat{V}$ to be the sum of the eigenspaces corresponding to the eigenvalues $\mu \leq \underline{\mu}$ (so that $V = \hat{V} \oplus V_0$) and $\hat{W} = V_0 \oplus W$. Therefore $X = \hat{V} \oplus \hat{W}$.

Also let e be any unit vector in $V_0 \subset \hat{W}$. First consider the behaviour of I_μ on the complementary subspaces $\hat{V}, \hat{W}$ when μ varies in the larger interval

$$\underline{\mu} < \mu < \mu_0 - m_0.$$

One can check that $I_\mu \leq 0$ on $\hat{V}$ (using (H0) and the variational characterization of $\underline{\mu}$) and that, for suitable $\beta > 0$ and $\rho > 0$, $I_\mu \geq \beta$ on $S_\rho \cap \hat{W}$ (using (H3), and in particular the definition of m_0).

Moreover, when $\mu_0 - m_\infty < \mu$, we have (using (H3), and in particular the definition of m_∞), that

$$I_\mu(u) \to -\infty \quad \text{as} \quad \|u\| \to \infty \quad \text{with} \quad u \in V.$$

Using the above Remark, we see that for $\mu_0 - m_\infty < \mu < \mu_0 - m_0$, all conditions of Theorem B are satisfied except the verification of the (PS) condition for I_μ. However, this can be checked making use of results of de Figueiredo ([DF], Lemma 6.3); see [C2].

Corollary. *Assume that m satisfies (H0), (H1) and (H3) with $m_0 = 0$ and $0 < m_\infty = a$. Then Σ_0 is the (open, or closed, or semiopen) interval of endpoints $\mu_0 - a$ and μ_0.*

Example. Suppose that for fixed $x \in \Omega$, $m(x, s)$ is increasing for $s > 0$ and decreasing for $s < 0$. Then $m_\infty = a$ (with $a = \sup_{(x,s) \in \Omega \times \mathbb{R}} m(x, s)$).

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